Pergamon

Int. J. Solids Structures Vol. 31, No. 9, pp. 1291–1308, 1994

0020–7683/94 $6.00 + .00

SHEAR BAND FORMULATIONS IN FINITE STRAIN ELASTOPLASTICITY

LÁSZLÓ SZABÓ
Department of Applied Mechanics, Technical University of Budapest, H-1521 Budapest, Hungary

(*Received* 20 *March* 1992; *in revised form* 22 *September* 1993)

Abstract—In the present paper, the shear band localization is studied for the case of large elastic–plastic deformation. In the first part, Rice's formula for the plastic hardening modulus is unfolded to cover five constitutive relations. The first four are based on the general nonassociative flow rule and on different objective stress rates, and the last one is the simple J_2 corner theory with the von Mises yield function. Moreover, some useful expressions for the acoustic tensor of these models are presented. In the second part, the explicit expressions for the shear band orientation and the plastic hardening modulus are given for the Jaumann rate formulation of the Cauchy stress tensor. These expressions are valid for a deviatoric associative flow rule, and it assumed that the stress tensor and the unit outward to the plastic and yield surface are coaxial. In addition, it has been proved that in the case of the Jaumann–Cauchy formulation the vector normal to the critical plane of localization is perpendicular to the direction of the second component of the unit deviatoric stress tensor.

1. INTRODUCTION

In recent years there has been active research work in the field of shear band localization. The basic principles were discussed by Hadamard (1903), Thomas (1961), Hill (1962), and Mandel (1966), in connection with the theory of bifurcation and localization corresponding to stationary acceleration waves. Later, Rice (1976) has presented a comprehensive discussion of these studies and given an explicit expression for the plastic hardening modulus in the small deformation range. Based on Rice's work, the general formulation of localization of deformation into shear bands in the small deformation range can be considered well established, and it was applied to predict the orientation of shear bands within various types of material models (see e.g. Bardet, 1990; Bigoni and Hueckel, 1990, 1991; Ottosen and Runesson, 1991a; Runesson *et al.*, 1991).

A standard method of calculating the critical shear band orientation and critical plastic hardening modulus is based on the vanishing of the determinant of the acoustic tensor, which is derived from the incremental constitutive stiffness tensor. This condition yields a set of plastic hardening moduli from which the critical value is obtained in a maximization procedure.

However, in several cases, for example when the critical hardening modulus is the same order as or smaller than the initial stress level, as was pointed out by Bazant (1988), Duszek and Perzyna (1991), the small deformation formulation is not satisfactory. Moreover, according to the literature (see e.g. Zbib and Aifantis, 1988b; Zbib, 1989, 1991, 1993; Tvergaard and Van der Giessen, 1991), the noncoaxiality between the stress and plastic stretching tensors has significance implications to the localization problem. Thus, when the noncoaxiality appears due to the corotational stress rates (e.g. Dafalias, 1983, 1985a, b; Dafalias and Aifantis, 1990; Loret, 1983; Van der Giessen, 1991; Zbib and Aifantis, 1988a; Zbib, 1991, 1993; Paulun and Pecherski, 1987), it is also necessary to consider large deformation. In addition, from a theoretical point of view, it is important to examine the shear band localization for large deformations. This was first analyzed by Rudnicki and Rice (1975), Hill and Hutchinson (1975), Young (1976) and Asaro and Rice (1977), and later in several studies discussed by Hutchinson and Tvergaard (1981), Mear and Hutchinson (1985), Needleman and Rice (1978), Needleman (1979), Rice and Rudnicki (1980), Tvergaard and Van der Giessen (1991) and Yatomi *et al.* (1989). In most of these studies, the presented shear band formulation neglected the higher order terms of the stress component divided by an elastic modulus or examined only simple loading cases with specially chosen coordinate systems.

The few cases where higher order stress components were considered were restricted in either the material model or in the loading cases discussed [see Zbib and Aifantis (1988b) (rigid–plastic material model), Hill and Hutchinson (1975), Young (1976) and Needleman (1979) (plane strain tension and compression of incompressible material) and recently Bardet (1991) (plane strain compression of compressible material), Zbib (1991, 1993), Bigoni and Hueckel (1993)]. To obtain a clear picture, the need arises to investigate the Rice's formulation for the plastic hardening modulus to the case of large elastic–plastic deformations.

The aim of this paper is to present some explicit expressions for the plastic hardening modulus for five commonly used elastoplastic models for large deformations. These models contain three objective stress rates (the Jaumann rate of the Cauchy and the Kirchhoff stress tensor, the Lie derivative of the Kirchhoff stress tensor and the corotational rate of the Kirchhoff stress tensor including the plastic spin), which have recently been used in the finite strain plasticity. The first four models are based on the general nonassociative flow rule, whose related incremental modulus tensor, operating on the rate of deformation tensor gives the stress rate in the model. In addition, the J_2 corner theory with the simple associative von Mises yield function, is discussed.

In the first part of the paper their constitutive relations are summarized. In the second part the acoustic tensor for these models are presented and from their determinants the plastic hardening moduli are evaluated. In addition, some useful formulations are given to the analysis of acceleration waves.

In the last part, the Jaumann rate formulation for the Cauchy stress is to be examined in the context of shear band localization. In this analysis it is assumed that the unit outward normals to the plastic potential and yield surface are coaxial. In this case the explicit expressions for the plastic hardening modulus and the shear band orientation are presented for a deviatoric associative flow rule.

In reference to notation, tensors will be denoted by bold-face characters, the order of which is indicated in the text. The tensor product is denoted by $\otimes$, and the following symbolic operations apply $\mathbf{g}\cdot\mathbf{n} = g_i n_i$, $(\mathbf{A}\cdot\mathbf{n})_i = A_{ij}n_j$, $(\mathbf{A}\cdot\mathbf{B})_{ij} = A_{ik}B_{kj}$, $\mathbf{A}{:}\mathbf{B} = A_{ij}B_{ij}$ and $(\mathbf{C}{:}\mathbf{A})_{ij} = C_{ijkl}A_{kl}$ with the summation convention over repeated indices. The superposed dot denotes the material time derivative or rate, and superscripts T and -1 denote transpose and inverse, and the prefix tr indicates the trace.

2. CONSTITUTIVE RELATIONS

The Jaumann rate formulations for the Cauchy and the Kirchhoff stress tensors

Consider a rate-independent, isotropic, homogeneous elasto-plastic material with a nonassociated flow rule and smooth plastic potential and yield surfaces. The widely used form of the constitutive relations in the finite deformation range is expressed in terms of a relation between the Jaumann rate of the Cauchy stress $\overset{\nabla}{\boldsymbol{\sigma}}$ and the rate of deformation $\mathbf{D}$ (see Rice, 1976; Rice and Rudnicki, 1980; Bardet, 1990). This is given by:

$$\overset{\nabla}{\boldsymbol{\sigma}} = \mathbf{C}^{ep}{:}\mathbf{D}, \tag{1}$$

where $\mathbf{C}^{ep}$ is the tensor of incremental elasto-plastic moduli. It may be written in the form:

$$\mathbf{C}^{ep} = \mathbf{C}^{e} - \frac{\mathbf{C}^{e}{:}\mathbf{P}\otimes\mathbf{Q}{:}\mathbf{C}^{e}}{H+\mathbf{Q}{:}\mathbf{C}^{e}{:}\mathbf{P}}, \tag{2}$$

where H is the plastic hardening modulus, and $\mathbf{P}$ and $\mathbf{Q}$ are the unit outward normals to the plastic potential and yield surfaces, respectively. Here $\mathbf{C}^{e}$ is the fourth-order isotropic elasticity tensor is defined as:

$$\mathbf{C}^e = 2G\mathbf{I} + \lambda\boldsymbol{\delta} \otimes \boldsymbol{\delta}, \tag{3}$$

where G and λ are the Lamé's constants and $\mathbf{I}$ and $\boldsymbol{\delta}$ are fourth-order and second-order unit tensors, respectively.

It is important to note that the elastic part of elastic–plastic constitutive tensor (2) relates a simple hypoelastic model which is not realistic in finite strain elasticity as pointed out by Simo and Pister (1984) and it is not accepted in many modern elastoplasticity formulations (see Simo, 1985, 1988). However, in order to compare the presented results with some others (e.g. Hill and Hutchinson, 1975; Rudnicki and Rice, 1975; Hutchinson and Tvergaard, 1981), a linear, isotropic elastic model [eqn (3)] is used in the present paper.

Choosing the Jaumann rate of the Kirchhoff stress tensor $\overset{\nabla}{\boldsymbol{\tau}}$ (see e.g. Asaro and Rice, 1977; Peirce, 1983; Prévost, 1984; Duszek and Perzyna, 1991; Tvergaard *et al.*, 1981), an alternative form of the constitutive equation (1) is given by

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathbf{C}^{ep}\mathbin{:}\mathbf{D}. \tag{4}$$

Corotational rate formulation including the plastic spin

There are several different suggestions for the corotational rate other than the Jaumann rate. One group of them uses the kinematic hardening model and the plastic spin concept. In this case the corotational stress rate for the Kirchhoff stress tensor is given by the expression:

$$\overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \boldsymbol{\omega}\cdot\boldsymbol{\tau} + \boldsymbol{\tau}\cdot\boldsymbol{\omega}. \tag{5}$$

In the above equation the spin tensor $\boldsymbol{\omega}$ is defined by:

$$\boldsymbol{\omega} = \mathbf{W} - \mathbf{W}^p, \tag{6}$$

where $\mathbf{W}$ is the vorticity and $\mathbf{W}^p$ is the plastic spin tensor. For the plastic spin various models have been proposed (e.g. Dafalias, 1983, 1985a, b; Loret, 1983; Zbib and Aifantis, 1988a; Voyiadjis and Kattan, 1989; Dafalias and Aifantis, 1990), which are outlined by Van der Giessen (1989), Voyiadjis and Kattan (1991), and Tvergaard and Van der Giessen (1991). One suggestion for the plastic spin within the framework of Mandel's theory is expressed as:

$$\mathbf{W}^p = \rho(\boldsymbol{\alpha}\cdot\mathbf{D}^p - \mathbf{D}^p\cdot\boldsymbol{\alpha}), \tag{7}$$

where $\boldsymbol{\alpha}$ is the back stress and $\mathbf{D}^p$ is the plastic part of the rate of deformation tensor.

Various plastic spin models can be obtained by assuming different expressions for the coefficient ρ (see e.g. Paulun and Pecherski, 1987; Zbib and Aifantis, 1988a; Dafalias and Aifantis, 1990; Van der Giessen *et al.*, 1992). In the simplest case, when $\rho = 0$, the corotational rate becomes identical to the Jaumann rate.

Using the corotational rate (7) we obtain the following constitutive relation:

$$\overset{\circ}{\boldsymbol{\tau}} = \mathbf{C}^{ep}\mathbin{:}\mathbf{D}. \tag{8}$$

The Lie derivative formulation for the Kirchhoff stress tensor

In the large strain plasticity formulations or in the constitutive modelling of elastoplastic materials in the large strain range, the convected (or Lie) derivations are frequently used. The Lie derivative of the contravariant Kirchhoff stress tensor (the so-called Oldroyd derivative) is given by:

$$\mathbf{L}_v\boldsymbol{\tau} = \dot{\boldsymbol{\tau}} - \mathbf{L}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\mathbf{L}^{\mathrm{T}}, \tag{9}$$

where $\mathbf{L}$ is the velocity gradient. Using this stress rate, the constitutive equation becomes, on the basis of Szabó (1988, 1992),

$$\mathbf{L}_v\boldsymbol{\tau} = \mathbf{E}^{ep}:\mathbf{D}, \tag{10}$$

where

$$\mathbf{E}^{ep} = \mathbf{C}^{e} - \frac{(\mathbf{C}^{e}:\mathbf{P})\otimes(\mathbf{Q}:\mathbf{C}^{e}+\mathbf{Q}\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\mathbf{Q})}{H+\mathbf{Q}:\mathbf{C}^{e}:\mathbf{P}}. \tag{11}$$

More recently, this type of constitutive relation has been discussed by Needleman and Ortiz (1991), Duszek and Perzyna (1991) and Szabó and Balla (1989).

The Jaumann rate formulation for the J_2 *corner theory*

The J_2 phenomenological corner theory was proposed by Christoffersen and Hutchinson (1979) on the basis of the J_2 deformation (or Hencky–Nadai) theory. The constitutive relation of this theory can be expressed as:

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathbf{M}^{ep}:\mathbf{D}, \tag{12}$$

where the constitutive tensor can be written in the following form:

$$\mathbf{M}^{ep} = \frac{2Gh_1}{1+h_1}\mathbf{I} + \left(K - \frac{2Gh_1}{3(1+h_1)}\right)\boldsymbol{\delta}\otimes\boldsymbol{\delta} - 2G\left(\frac{1}{1+h} - \frac{1}{1+h_1}\right)\mathbf{p}\otimes\mathbf{p}, \tag{13}$$

where K is the bulk modulus, and using the deviatoric Kirchhoff stress $\boldsymbol{\tau}'$ the second-order tensor $\mathbf{p}$ is defined by $\boldsymbol{\tau}'/(\boldsymbol{\tau}':\boldsymbol{\tau}')^{1/2}$ as the unit normal of the Von Mises yield surface.

In the above expression for $\mathbf{M}^{ep}$, the scalar parameters h and h_1 are taken as

$$h = \frac{H_t}{3G}g(\phi)[1-l(\phi)\tan\phi], \quad h_1 = \frac{H_s}{3G}g(\phi)[1+l(\phi)\tan^{-1}\phi], \tag{14}$$

where H_t and H_s are, respectively, the tangent and the secant moduli of the uniaxial stress–plastic strain curve, and the definitions of the angle ϕ and the $g(\phi)$ and $l(\phi)$ transition functions can be found in the papers of Christoffersen and Hutchinson (1979) or Tvergaard *et al.* (1981).

It is important to note that if the angle ϕ is dependent on $\mathbf{D}$, in eqn (14), the shear band analysis is more difficult and the method of the localization, using this paper, cannot be applied. In the case when ϕ is independent on $\mathbf{D}$, eqn (12) is an approximation of the constitutive equation proposed by Christoffersen and Hutchinson (1979).

3. SHEAR BAND FORMULATIONS

The general framework for the shear band localization analysis has been given by Rudnicki and Rice (1975), Hill and Hutchinson (1975) and Rice (1976) and was discussed, for example, by Rice and Rudnicki (1980), Hutchinson and Tvergaard (1981), Zbib and Aifantis (1988b) and Bigoni and Hueckel (1991). Here we will briefly summarize some necessary formulas. The outline given below follows Rice and Rudnicki (1980).

The velocity gradient inside the shear band is defined by

$$\mathbf{L} = \mathbf{L}^{o} + \mathbf{g}\otimes\mathbf{n}, \tag{15}$$

where $\mathbf{L}^{o}$ is the velocity gradient outside the shear band and $\mathbf{g}\otimes\mathbf{n}$ is the jump of the velocity gradient which is a function only of distance across a planar band and vanishes outside the band. Here $\mathbf{n}$ is the unit vector normal to the shear band and $\mathbf{g}$ is some vector which depends only on the distance across the band and which is zero outside of it.

Furthermore, there is the requirement of continuing equilibrium. It reads as

$$\mathbf{n}\cdot(\dot{\mathbf{s}}-\dot{\mathbf{s}}^{\circ}) = 0, \tag{16}$$

where $\mathbf{s}$ and $\dot{\mathbf{s}}^{\circ}$ are the nominal stress rates within the band and outside of it, respectively. When the reference state coincides with the current state, then the Cauchy stress, as remarked by Rudnicki and Rice (1975), has the same form as

$$\mathbf{n}\cdot(\dot{\boldsymbol{\sigma}}-\dot{\boldsymbol{\sigma}}^{\circ}) = 0. \tag{17}$$

Using the relationship between the nominal and the Kirchhoff stress rates $\dot{\mathbf{s}} = \dot{\boldsymbol{\tau}}-\mathbf{L}\cdot\boldsymbol{\tau}$, the equilibrium eqn (16) for the Kirchhoff stress rate can be rewritten (Duszek and Perzyna, 1991)

$$\mathbf{n}\cdot(\dot{\boldsymbol{\tau}}-\dot{\boldsymbol{\tau}}^{\circ})-(\boldsymbol{\tau}\cdot\mathbf{n})(\mathbf{g}\cdot\mathbf{n}) = 0. \tag{18}$$

We note that in choosing the reference state the Cauchy and the Kirchhoff stress tensors are identical, but their rates are different.

For the case of a continuous bifurcation, in which the constitutive response remains continuous at the inception of localization, by substitution of the constitutive relation into eqn (17) [or (18), respectively] and use of the expression in eqn (15), one obtains:

$$\mathbf{B}\cdot\mathbf{g} = 0. \tag{19}$$

The necessary condition for the localization is that a solution other than $\mathbf{g} \equiv 0$ exists:

$$\det(\mathbf{B}) = 0. \tag{20}$$

Here the second-order tensor $\mathbf{B}$ (the so-called acoustic tensor) for the different constitutive relations can be given in the following general form:

$$\mathbf{B} = \mathbf{B}^{e}+\gamma\mathbf{x}\otimes\mathbf{y}, \tag{21}$$

where

$$\gamma = -(H+\mathbf{P}:\mathbf{C}^{e}:\mathbf{Q})^{-1}, \tag{22}$$

and the elastic acoustic tensor $\mathbf{B}^{e}$ is given by

$$\mathbf{B}^{e} = \mathbf{n}\cdot\mathbf{C}^{e}\cdot\mathbf{n}+\mathbf{A}. \tag{23}$$

The second order tensor $\mathbf{A}$ in eqn (23) represents the stress rate effect, and the vectors $\mathbf{x}$ and $\mathbf{y}$ are dependent on $\mathbf{n}$ for the different constitutive models given below.

Remark 1

In the analysis of the acceleration waves an important area is associated with examination of the speeds of propagation. In practice, it means to give a solution of the eigenvalue problem for the acoustic tensor. Because the wave speeds are related to the formation of shear band, we derive some useful relation. The characteristic equation for the eigenvalues of $\mathbf{B}$ can be given by:

$$\kappa^{3}-\mathrm{I}_{\mathbf{B}}\kappa^{2}+\mathrm{II}_{\mathbf{B}}\kappa-\det\mathbf{B} = 0, \tag{24}$$

where $\mathrm{I}_{\mathbf{B}}$ and $\mathrm{II}_{\mathbf{B}}$ are the first and second scalar invariants of the tensor $\mathbf{B}$. In order to calculate these quantities we need some simple relationships. The second scalar invariant of the sum of two second-order tensor can be expressed as:

$$\mathrm{II}_{(\mathbf{A}+\mathbf{C})} = \mathrm{II}_{\mathbf{A}} + \mathrm{II}_{\mathbf{C}} + \mathrm{I}_{\mathbf{A}}\mathrm{I}_{\mathbf{C}} - \mathrm{tr}\,(\mathbf{CA}). \tag{25}$$

The calculation of determinant **B**, using the Cayley–Hamilton equation, gives the well-known expression:

$$\det(\mathbf{B}) = \tfrac{1}{3}\mathrm{tr}\,\mathbf{B}^3 + \tfrac{1}{6}(\mathrm{tr}\,\mathbf{B})^3 - \tfrac{1}{2}\mathrm{tr}\,\mathbf{B}\,\mathrm{tr}\,\mathbf{B}^2. \tag{26}$$

For determinant of the sum of two second-order tensors, using eqn (26), we obtain:

$$\det(\mathbf{A}+\mathbf{C}) = \det\mathbf{A} + \det\mathbf{C} + \mathrm{I}_{\mathbf{A}}\mathrm{II}_{\mathbf{C}} + \mathrm{I}_{\mathbf{C}}\mathrm{II}_{\mathbf{A}} - (\mathrm{I}_{\mathbf{A}} + \mathrm{I}_{\mathbf{C}})\,\mathrm{tr}\,(\mathbf{CA}) + \mathrm{tr}\,(\mathbf{CA}^2) + \mathrm{tr}\,(\mathbf{C}^2\mathbf{A}). \tag{27}$$

Then, with the use of eqns (25) and (27) we can derive the scalar invariants of **B**:

$$\left.\begin{aligned}
\mathrm{I}_{\mathbf{B}} &= \mathrm{I}_{\mathbf{B}^e} + \gamma\mathbf{x}\cdot\mathbf{y} = e_1 + \gamma p_1 \\
\mathrm{II}_{\mathbf{B}} &= \mathrm{II}_{\mathbf{B}^e} + \gamma\{(\mathbf{x}\cdot\mathbf{y})\mathrm{I}_{\mathbf{B}^e} - \mathbf{y}\cdot\mathbf{B}^e\cdot\mathbf{x}\} = e_2 + \gamma p_2 \\
\det\mathbf{B} &= \det\mathbf{B}^e + \gamma\{(\mathbf{x}\cdot\mathbf{y})\mathrm{II}_{\mathbf{B}^e} - (\mathbf{y}\cdot\mathbf{B}^e\cdot\mathbf{x})\mathrm{I}_{\mathbf{B}^e} + \mathbf{y}\cdot(\mathbf{B}^e)^2\cdot\mathbf{x}\} = e_3 + \gamma p_3.
\end{aligned}\right\} \tag{28}$$

For the characteristic eqn (24) to have three real roots, its discriminant D must be negative, i.e.

$$108D = -(\mathrm{I}_{\mathbf{B}}\mathrm{II}_{\mathbf{B}})^2 + 4(\mathrm{II}_{\mathbf{B}})^3 - 18\mathrm{I}_{\mathbf{B}}\mathrm{II}_{\mathbf{B}}\det\mathbf{B} + 4\mathrm{I}_{\mathbf{B}}^3\det\mathbf{B} + 27(\det\mathbf{B})^2 < 0. \tag{29}$$

Following Loret and Harireche (1991), using the decomposition (28) of each scalar invariant, this condition can be expressed as a fourth degree polynomial with respect to H:

$$D(H) = d_4H^4 + d_3H^3 + d_2H^2 + d_1H + d_0 < 0, \tag{30}$$

where the parameters d_i ($i = 0$–4) are given in Appendix A. □

From the condition of localization (20), by combining eqns $(28)_3$ and (22), we may now formulate the expression for the plastic hardening modulus:

$$H = -\mathbf{P}{:}\mathbf{C}^e{:}\mathbf{Q} + \frac{(\mathbf{x}\cdot\mathbf{y})\mathrm{II}_{\mathbf{B}^e} - (\mathbf{y}\cdot\mathbf{B}^e\cdot\mathbf{x})\mathrm{I}_{\mathbf{B}^e} + \mathbf{y}\cdot(\mathbf{B}^e)^2\cdot\mathbf{x}}{\det\mathbf{B}^e}. \tag{31}$$

This equation shows that when the elastic acoustic tensor is singular the plastic hardening modulus cannot be expressed as pointed out by Rice (1976).

The Jaumann rate formulation for the Cauchy stress tensor

For the constitutive relation (1) the tensor **B** is defined (see e.g. Rice, 1976; Rice and Rudnicki, 1980; Bardet, 1991) by:

$$\mathbf{B} = \mathbf{n}\cdot\mathbf{C}^e\cdot\mathbf{n} + \mathbf{A} - \frac{\mathbf{a}\otimes\mathbf{b}}{H + \mathbf{P}{:}\mathbf{C}^e{:}\mathbf{Q}}, \tag{32}$$

where

$$\mathbf{A} = \tfrac{1}{2}[(\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\delta} + \boldsymbol{\sigma}\cdot\mathbf{n}\otimes\mathbf{n} - \mathbf{n}\otimes\mathbf{n}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}], \tag{33}$$

and using eqns (2) and (3), the vectors **a** and **b** are defined as:

$$\left.\begin{aligned} \mathbf{a} &= \mathbf{n}\cdot(\mathbf{C}^e:\mathbf{P}) = 2G\mathbf{n}\cdot\mathbf{P}+\lambda\mathbf{n}\,\mathrm{tr}\,\mathbf{P} \\ \mathbf{b} &= (\mathbf{Q}:\mathbf{C}^e)\cdot\mathbf{n} = 2G\mathbf{Q}\cdot\mathbf{n}+\lambda\mathbf{n}\,\mathrm{tr}\,\mathbf{Q} \end{aligned}\right\}. \tag{34}$$

Insertion of the elasticity tensor (3) into eqn (32), after some manipulations we obtain:

$$\mathbf{B} = (G+\tfrac{1}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\delta}+(G+\lambda)\mathbf{n}\otimes\mathbf{n}-\frac{\mathbf{a}\otimes\mathbf{b}}{H+\mathbf{P}:\mathbf{C}^e:\mathbf{Q}}-\tfrac{1}{2}\boldsymbol{\sigma}+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\mathbf{n}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{n}\cdot\boldsymbol{\sigma}). \tag{35}$$

Defining a stress tensor $\tilde{\boldsymbol{\sigma}} = \frac{1}{2}\boldsymbol{\sigma}$, then, from condition (20), using eqn (31), the plastic hardening modulus is determined as:

$$\begin{aligned} H = {} & -2G\mathbf{P}:\mathbf{Q}-\lambda\,\mathrm{tr}\,\mathbf{P}\,\mathrm{tr}\,\mathbf{Q} \\ & +k_1\{(\mathbf{a}\cdot\mathbf{b})[(2G+\lambda)(G+2\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\sigma}})-(\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n})\mathrm{I}_{\tilde{\sigma}}+\mathrm{II}_{\tilde{\sigma}}+\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}^2\cdot\mathbf{n}] \\ & -(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})[(G+\lambda)(G+\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\sigma}})+\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}^2\cdot\mathbf{n}] \\ & -(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n})(2G+\lambda-\mathrm{I}_{\tilde{\sigma}})+(\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n})(2\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}-\lambda-\mathrm{I}_{\tilde{\sigma}}) \\ & +(\mathbf{a}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{b})(2G+\lambda+\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\sigma}}) \\ & +(\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\tilde{\boldsymbol{\sigma}}^2\cdot\mathbf{n})-(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\tilde{\boldsymbol{\sigma}}^2\cdot\mathbf{n})-(\mathbf{a}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n})(\mathbf{b}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}) \\ & +\mathbf{a}\cdot\tilde{\boldsymbol{\sigma}}^2\cdot\mathbf{b}\}, \end{aligned} \tag{36}$$

where

$$\frac{1}{k_1} = (2G+\lambda)[(G+\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\sigma}})(G+2\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n})+\mathrm{II}_{\tilde{\sigma}}+\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}^2\cdot\mathbf{n}]. \tag{37}$$

Here $\mathrm{I}_{\tilde{\sigma}}$ and $\mathrm{II}_{\tilde{\sigma}}$ are the first and second scalar invariants of the stress tensor $\tilde{\boldsymbol{\sigma}}$, respectively. In fact, if one substitutes $\tilde{\boldsymbol{\sigma}} = 0$ into eqn (36) the well-known formulation for the plastic hardening modulus in the small deformation case results (see e.g. Rice, 1976; Bardet, 1990; Bigoni and Hueckel, 1991; Prévost, 1984).

Remark 2

The determinant of the elastic acoustic tensor is defined by eqn (37). From this equation it is seen that one eigenvalue defined by $2G+\lambda$ with the eigenvector $\mathbf{n}$, corresponds to the longitudinal wave speed. The remaining eigenvalues are given by:

$$\kappa_{2,3} = \tfrac{1}{2}\{2G+3\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\sigma}}\pm\sqrt{(\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}+\mathrm{I}_{\tilde{\sigma}})^2-4(\mathrm{II}_{\tilde{\sigma}}+\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}^2\cdot\mathbf{n})}\}. \tag{38}$$

Because $\mathbf{B}^e$ is nonsymmetric, the eigenvectors become nonorthogonal. It is easy to prove that a complex conjugate root never occurs from the above expression. □

Proposition 1

The eigenvalues of the elastic acoustic tensor in the case of the Jaumann–Cauchy formulation are always real. In other words, for the discriminant in eqn (38),

$$D = (\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}+\mathrm{I}_{\tilde{\sigma}})^2-4(\mathrm{II}_{\tilde{\sigma}}+\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}^2\cdot\mathbf{n}) \geqslant 0 \tag{39}$$

always holds.

Proof

Define the deviatoric normal and shear stress in the following form:

$$\hat{\sigma}_n = \mathbf{n}\cdot\hat{\mathbf{S}}\cdot\mathbf{n}, \quad \hat{\tau}_n = \sqrt{\mathbf{n}\cdot\hat{\mathbf{S}}^2\cdot\mathbf{n}-(\mathbf{n}\cdot\hat{\mathbf{S}}\cdot\mathbf{n})^2}, \tag{40}$$

where $\hat{\mathbf{S}} = \mathbf{s}/(\mathbf{s}:\mathbf{s})^{1/2}$ is the unit deviatoric stress. Here $\mathbf{s}$ is the deviatoric Cauchy stress. Using these quantities we may now reformulate the discriminant (39) as:

$$D = 2J_2(\tfrac{1}{2}-\hat{\tau}_n^2-\tfrac{3}{4}\hat{\sigma}_n^2) \geqslant 0, \tag{41}$$

where J_2 is the second invariant of the deviatoric stress tensor. For $D = 0$, this function represents an ellipse in the $(\hat{\sigma}_n, \hat{\tau}_n)$ plane, namely:

$$\hat{\tau}_n^2+\tfrac{3}{4}\hat{\sigma}_n^2-\tfrac{1}{2} = 0. \tag{42}$$

The stress points, which are satisfied by the condition (41), are located inside or on this ellipse. We shall now prove that the ellipse (42) is the enveloping curve of all the largest Mohr circles.

The largest Mohr circle on the $(\hat{\sigma}_n, \hat{\tau}_n)$ plane is given by:

$$\hat{\tau}_n^2+\left(\hat{\sigma}_n-\frac{\hat{S}_1+\hat{S}_3}{2}\right)^2 = \left(\frac{\hat{S}_1-\hat{S}_3}{2}\right)^2, \tag{43}$$

or in an alternative form

$$\hat{\tau}_n^2+\hat{\sigma}_n^2+\hat{\sigma}_n\hat{S}_2+\hat{S}_2^2-\tfrac{1}{2} = 0, \tag{44}$$

where $\hat{S}_1 > \hat{S}_2 > \hat{S}_3$ are the principal values of tensor $\hat{\mathbf{S}}$, which satisfy:

$$\hat{S}_1+\hat{S}_2+\hat{S}_3 = 0, \quad \hat{S}_1^2+\hat{S}_2^2+\hat{S}_3^2 = 1. \tag{45}$$

By differentiating eqn (44) with respect to $\hat{S}_2$, we obtain $\hat{S}_2 = -\hat{\sigma}_n/2$, which substitutes into eqn (44), the result is identical with eqn (42). □

The Jaumann rate formulation for the Kirchhoff stress tensor

In the case of the constitutive equation (4) the tensor $\mathbf{B}$ is found (see Asaro and Rice, 1977; Peirce, 1983; Duszek and Perzyna, 1991) to be:

$$\mathbf{B} = \mathbf{n}\cdot\mathbf{C}^e\cdot\mathbf{n}+\mathbf{A}-\frac{\mathbf{a}\otimes\mathbf{b}}{H+\mathbf{P}:\mathbf{C}^e:\mathbf{Q}}, \tag{46}$$

where

$$\mathbf{A} = \tfrac{1}{2}[(\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})\boldsymbol{\delta}-\boldsymbol{\tau}\cdot\mathbf{n}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{n}\cdot\boldsymbol{\tau}-\boldsymbol{\tau}]. \tag{47}$$

Similarly to the previous case, a substitution of tensors (3) into eqn (46) then gives:

$$\mathbf{B} = (G+\tfrac{1}{2}\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})\boldsymbol{\delta}+(G+\lambda)\mathbf{n}\otimes\mathbf{n}-\frac{\mathbf{a}\otimes\mathbf{b}}{H+\mathbf{P}:\mathbf{C}^e:\mathbf{Q}}-\tfrac{1}{2}\boldsymbol{\tau}-\tfrac{1}{2}(\boldsymbol{\tau}\cdot\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}\cdot\boldsymbol{\tau}). \tag{48}$$

Defining also a stress tensor $\tilde{\boldsymbol{\tau}} = \frac{1}{2}\boldsymbol{\tau}$ and using eqns (31) and (48) the expression for H is given by:

$$\begin{aligned} H = {} & -2G\mathbf{P}:\mathbf{Q}-\lambda\,\mathrm{tr}\,\mathbf{P}\,\mathrm{tr}\,\mathbf{Q} \\ & +k_2\{(\mathbf{a}\cdot\mathbf{b})[(2G+\lambda)(G+\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}}) \\ & +(\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})(\lambda+\mathrm{I}_{\tilde{\tau}})+\mathrm{II}_{\tilde{\tau}}-3\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{n}] \\ & -(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})[(G+\lambda)(G+\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}})-\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{n}] \\ & -[(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})+(\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})](\lambda+\mathrm{I}_{\tilde{\tau}}) \\ & +(\mathbf{a}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{b})(2G+\lambda-\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}}) \\ & +(\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{n})+(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{n})+3(\mathbf{a}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})(\mathbf{b}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}) \\ & +\mathbf{a}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{b}\}, \end{aligned} \tag{49}$$

where

$$\frac{1}{k_2} = G(2G+\lambda)(G+2\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}})$$
$$+2(\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})(G+\lambda)(\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}})+G\lambda\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}$$
$$+(\lambda-2G+3\mathrm{I}_{\tilde{\tau}}-2\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})(\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{n})$$
$$+(2G+\lambda-\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})\mathrm{II}_{\tilde{\tau}}-3\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}^3\cdot\mathbf{n}-\det(\tilde{\boldsymbol{\tau}}). \quad (50)$$

Remark 3

Note that the difference in the tensor **B** between the Jaumann rate formulation for the Cauchy (35) and the Kirchhoff stress tensors (48) appears only in the last two terms. In eqn (35) these terms contain the antisymmetric part of $\boldsymbol{\sigma}\cdot\mathbf{n}\otimes\mathbf{n}$, in eqn (48) the symmetric part of $\boldsymbol{\tau}\cdot\mathbf{n}\otimes\mathbf{n}$. □

Corotational rate formulation for the Kirchhoff stress tensor including the plastic spin

Because the corotational stress rate (5) contains the plastic strain rate, the expression for the jump of this quantity is needed. According to work of Rice and Rudnicki (1980) the following expression for $\Delta\mathbf{D}^p$ can be obtained

$$\Delta\mathbf{D}^p = \mathbf{P}\left(\frac{\mathbf{b}\cdot\mathbf{g}}{H+\mathbf{P}:\mathbf{C}^e:\mathbf{Q}}\right). \quad (51)$$

Substituting eqn (8) into (18) and using eqns (5)–(7), (36) and (51) the tensor **B** becomes

$$\mathbf{B} = (G+\tfrac{1}{2}\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})\boldsymbol{\delta}+(G+\lambda)\mathbf{n}\otimes\mathbf{n}-\frac{(\mathbf{a}-\mathbf{d})\otimes\mathbf{b}}{H+\mathbf{P}:\mathbf{C}^e:\mathbf{Q}}-\tfrac{1}{2}\boldsymbol{\tau}-\tfrac{1}{2}(\boldsymbol{\tau}\cdot\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}\cdot\boldsymbol{\tau}), \quad (52)$$

where

$$\mathbf{d} = \rho\mathbf{n}\cdot(\boldsymbol{\alpha}\cdot\mathbf{P}\cdot\boldsymbol{\tau}-\mathbf{P}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\cdot\mathbf{P}+\boldsymbol{\tau}\cdot\mathbf{P}\cdot\boldsymbol{\alpha}). \quad (53)$$

The expression for the plastic hardening modulus H can be written in the same form as in the previous case, eqn (49), by making the following identification $\mathbf{a}\to\mathbf{a}-\mathbf{d}$. In a similar way, the corotational formulation for the Cauchy stress tensor can be obtained by using eqn (36).

Remark 4

It may be observed that the vector **d** (53) vanishes when $\boldsymbol{\alpha}$ and **P** are coaxial. In addition, the plastic hardening modulus usually is divided by the elastic shear modulus. In this case the stress tensors in the eqns (36) and (49) must also be divided by the shear modulus. However, it is of interest that in the case of the corotational formulation including the plastic spin in eqn (53), $\boldsymbol{\tau}$ is divided by G but $\boldsymbol{\alpha}$ is not. In this case the tensor $\boldsymbol{\alpha}$ is only multiplied by the parameter ρ. Since the magnitude of the parameter ρ, according to the literature (for example Dafalias, 1985a; Dafalias and Aifantis 1990; Loret, 1983; Paulun and Pecherski, 1987; Tvergaard and Van der Giessen, 1991; Van der Giessen *et al.*, 1992) usually is $\rho \gg 1/G$, the effect of the plastic spin in all of the terms in eqn (49) may be relatively significant. □

This model has recently been discussed by Zbib and Aifantis (1988b), Zbib (1991, 1993).

The Lie derivative formulation for the Kirchhoff stress tensor

In this case, from the constitutive equation (10) the result (see Duszek and Perzyna, 1991) becomes:

$$\mathbf{B} = \mathbf{n}\cdot\mathbf{E}^{ep}\cdot\mathbf{n}+(\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})\boldsymbol{\delta}; \tag{54}$$

or another new form:

$$\mathbf{B} = (G+\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})\boldsymbol{\delta}+(G+\lambda)\mathbf{n}\otimes\mathbf{n}-\frac{\mathbf{a}\otimes(\mathbf{b}+\mathbf{c})}{H+\mathbf{P}:\mathbf{C}^{e}:\mathbf{Q}}, \tag{55}$$

where **c** is defined by

$$\mathbf{c} = (\mathbf{Q}\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\mathbf{Q})\cdot\mathbf{n}. \tag{56}$$

Using the localization condition (20), the plastic hardening modulus is given by the expression:

$$\begin{aligned} H = {} & -2G\mathbf{P}:\mathbf{Q}-\lambda\,\mathrm{tr}\,\mathbf{P}\,\mathrm{tr}\,\mathbf{Q} \\ & +\{\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})(2G+\lambda+\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})-(\mathbf{a}\cdot\mathbf{n})[(\mathbf{b}+\mathbf{c})\cdot\mathbf{n}](G+\lambda)\} \\ & \times\{(G+\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})(2G+\lambda+\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})\}^{-1}. \end{aligned} \tag{57}$$

We note that the constitutive tensor defined in (11) is not symmetric even when $\mathbf{P} = \mathbf{Q}$. The unsymmetrical acoustic tensor (55) follows.

Remark 5

For the Lie derivative formulation of the Kirchhoff stress tensor (10) the tensor **B** is given in the same form as in the case of small deformations. In this case, the tensor **B** may be written in the general form:

$$\mathbf{B} = \alpha\boldsymbol{\delta}+\beta\mathbf{n}\otimes\mathbf{n}+\gamma\mathbf{x}\otimes\mathbf{y}, \tag{58}$$

where the β and γ parameters are the same for the small deformation and the large deformation cases. Moreover, the α parameter equals G, $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$ for the small deformation, and $\alpha = G+\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n}$, $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}+\mathbf{c}$ for the Lie derivative formulation. Substituting (58) into $(28)_3$ the determinant of the acoustic tensor becomes

$$\det(\mathbf{B}) = \alpha[(\alpha+\beta)(\alpha+\gamma\mathbf{x}\cdot\mathbf{y})-\beta\gamma(\mathbf{x}\cdot\mathbf{n})(\mathbf{y}\cdot\mathbf{n})]. \tag{59}$$

From eqn (59) and the characteristic equation of **B** it follows that one eigenvalue must be identical to:

$$\kappa_1 = \alpha, \tag{60}$$

as was firstly shown that by Hill (1962) for the large deformation, and by Loret *et al.* (1990) and Ottosen and Runesson (1991b) for the small deformation. The other two eigenvalues can be expressed as:

$$\kappa_{2,3} = \tfrac{1}{2}\{2\alpha+\beta+\gamma(\mathbf{x}\cdot\mathbf{y})\pm\sqrt{[\beta-\gamma(\mathbf{x}\cdot\mathbf{y})]^2+4\beta\gamma(\mathbf{x}\cdot\mathbf{n})(\mathbf{y}\cdot\mathbf{n})}\}. \tag{61}$$

According to this remark it would be easy to generalize the results presented by Loret *et al.* (1990). □

The Jaumann rate formulation of the J_2 *corner theory*

For the J_2 corner theory, the constitutive moduli $\mathbf{M}^{ep}$ (13) are inserted into eqn (46) instead of $\mathbf{C}^{ep}$, and we obtain:

$$\mathbf{B} = \left(\frac{Gh_1}{1+h_1} + \frac{1}{2}\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n}\right)\boldsymbol{\delta} + \left(K + \frac{Gh_1}{3(1+h_1)}\right)\mathbf{n}\otimes\mathbf{n}$$

$$-2G\left(\frac{1}{1+h} - \frac{1}{1+h_1}\right)\mathbf{n}\cdot\mathbf{p}\otimes\mathbf{p}\cdot\mathbf{n} - \tfrac{1}{2}\boldsymbol{\tau} - \tfrac{1}{2}(\boldsymbol{\tau}\cdot\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}\cdot\boldsymbol{\tau}). \quad (62)$$

Introduce two simple parameters as follows:

$$\xi = \frac{1}{h_1+1}, \quad \zeta = \frac{h_1}{h_1+1}.$$

With them we can define modified Lamé's parameters

$$G_1 = G\zeta, \quad \lambda_1 = K - \tfrac{2}{3}G_1.$$

Using these parameters the tensor **B** may be rewritten,

$$\mathbf{B} = (G_1 + \tfrac{1}{2}\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})\boldsymbol{\delta} + (\lambda_1 + G_1)\mathbf{n}\otimes\mathbf{n}$$

$$-2G\left(\frac{1}{1+h} - \xi\right)\mathbf{n}\cdot\mathbf{p}\otimes\mathbf{p}\cdot\mathbf{n} - \tfrac{1}{2}\boldsymbol{\tau} - \tfrac{1}{2}(\boldsymbol{\tau}\cdot\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}\cdot\boldsymbol{\tau}). \quad (63)$$

Following the same procedure as previously, the corresponding expression for the plastic hardening modulus h is given by

$$h = -1 + \frac{2Gk_3}{2G\xi k_3 + k_4}, \quad (64)$$

where

$$\begin{aligned}
k_3 = {} & (\mathbf{n}\cdot\mathbf{p}^2\cdot\mathbf{n})[(2G_1+\lambda_1)(G_1+\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}}) \\
& +(\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})(\lambda_1+\mathrm{I}_{\tilde{\tau}})+\mathrm{II}_{\tilde{\tau}}-3\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{n}] \\
& -(\mathbf{n}\cdot\mathbf{p}\cdot\mathbf{n})^2[(G_1+\lambda_1)(G_1+\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}})-\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{n}] \\
& -2(\mathbf{n}\cdot\mathbf{p}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{p}\cdot\boldsymbol{\tau}\cdot\mathbf{n})(\lambda_1+\mathrm{I}_{\tilde{\tau}}) \\
& +(\mathbf{n}\cdot\mathbf{p}\cdot\boldsymbol{\tau}\cdot\mathbf{p}\cdot\mathbf{n})(2G_1+\lambda_1-\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}}) \\
& +2(\mathbf{n}\cdot\mathbf{p}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{p}\cdot\boldsymbol{\tau}^2\cdot\mathbf{n})+3(\mathbf{n}\cdot\mathbf{p}\cdot\boldsymbol{\tau}\cdot\mathbf{n})^2+\mathbf{n}\cdot\mathbf{p}\cdot\boldsymbol{\tau}^2\cdot\mathbf{p}\cdot\mathbf{n},
\end{aligned}$$

$$\begin{aligned}
k_4 = {} & G_1(2G_1+\lambda_1)(G_1+2\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}}) \\
& +2(\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})(G_1+\lambda_1)(\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}-\mathrm{I}_{\tilde{\tau}})+G_1\lambda_1\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n} \\
& +(\lambda_1-2G_1+3\mathrm{I}_{\tilde{\tau}}-2\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})(\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}^2\cdot\mathbf{n}) \\
& +(2G_1+\lambda_1-\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}\cdot\mathbf{n})\mathrm{II}_{\tilde{\tau}}-3\mathbf{n}\cdot\tilde{\boldsymbol{\tau}}^3\cdot\mathbf{n}-\det(\tilde{\boldsymbol{\tau}}).
\end{aligned}$$

It is important to note that, the mathematical form of the expression for h in eqn (64) is basically different from the H expressions in eqns (36), (49) and (57). In these expressions, the divisor is the determinant of the elastic part of **B** [eqn (23)] for large elastic deformations. In eqn (64) the divisor contains both the elastic and the elastic–plastic parts of the determinant of tensor **B**.

Remark 6

Note that in the limit when $h_1 \to \infty$, then $\xi \to 0$ and $\zeta \to 1$ and the formulation (62) is reduced to the J_2 flow theory. In this case, because of G_1 and λ_1 being reduced to G and λ, the expression for the plastic hardening modulus (63) becomes identical to (49), in which the associated Mises flow rule ($\mathbf{P} = \mathbf{Q} = \mathbf{p}$ and tr $\mathbf{P}$ = tr $\mathbf{Q} = 0$) was assumed to be valid. Moreover, if in the expression (14) $g(\phi) = 1$ and $l(\phi) = 0$ the equation for the plastic hardening modulus (64) is reduced to the J_2 deformation theory. In fact, in this case the plastic hardening modulus is defined by $H = 2H_t/3$ [or $h = H_t/(3G)$]. □

Finally, we note that from the constitutive models discussed in this paper the co-rotational stress rate formulation and the J_2 corner theory (or vertex-type plasticity models) are shown to be significant effects on the shear band localization. This fact implies that a combination of these models may be interesting in a further analysis, and as it has recently been discussed by Zbib and Aifantis (1988b), Zbib (1991, 1993) and Zhu *et al.* (1992).

The critical hardening modulus corresponds to the solution of the constrained maximization problem:

$$\frac{H_{cr}}{2G} = \max_{\mathbf{n}} \left\{ \frac{H}{2G} \right\},$$

subject to $|\mathbf{n}| = 1$. Here $H/2G$ may be substituted by the expressions of eqns (36), (49), (57) and (63). In the general three-dimensional case the vector $\mathbf{n}$ can be assumed as $\mathbf{n}^T = [\cos\vartheta \cos\theta, \cos\vartheta \sin\theta, \sin\vartheta]$, where ϑ and θ are the spherical angles. Using this form of vector $\mathbf{n}$, the solution of the maximization problem becomes a high order (in $\tan^2\vartheta$ and $\tan^2\theta$) nonlinear equations system with two equations, which can be solved by a pure algebraic method.

4. EXAMPLES

In this section, as an illustration, the Jaumann rate formulations for the Cauchy stress tensor, eqn (36), with a deviatoric associative flow rule is investigated in detail. The assumption of deviatoric associativity amounts to postulating the unit tensor $\mathbf{P}$ and $\mathbf{Q}$ in the following forms (Loret *et al.*, 1990; Loret, 1992):

$$\mathbf{Q} = \cos\alpha\mathbf{S} + \frac{1}{\sqrt{3}}\sin\alpha\boldsymbol{\delta}, \quad \mathbf{P} = \cos\beta\mathbf{S} + \frac{1}{\sqrt{3}}\sin\beta\boldsymbol{\delta}, \tag{65}$$

where $0 \leqslant \beta \leqslant \alpha < \pi/2$ and $\mathbf{S}$ is the unit deviatoric stress which can be written in terms of the single scalar $l \in [0, \pi/3]$ called the Lode angle:

$$S_i = \sqrt{\frac{2}{3}}\cos\left[l - \frac{2}{3}(i-1)\pi\right]; \quad i \in [1, 3]. \tag{66}$$

Substituting eqn (65) into (34) and using the deviatoric normal stress $\hat{\sigma}_n$ and shear stress $\hat{\tau}_n$ [eqn (40)] on the plane of localization, the expression for the plastic hardening modulus eqn (36) can be rewritten in the following form:

$$\frac{H}{2G} = c_0 + \frac{2\cos\alpha\cos\beta[(\hat{\sigma}_n^2+\hat{\tau}_n^2)(\eta^2\hat{\sigma}_n^2+a_1\hat{\sigma}_n+a_2)+a_3\hat{\sigma}_n^3+a_4\hat{\sigma}_n^2+a_5\hat{\sigma}_n+a_6]}{(2+\mu)[\eta^2(\hat{\sigma}_n^2+\hat{\tau}_n^2)+2\eta^2\hat{\sigma}_n^2+3\eta\hat{\sigma}_n+1-\frac{1}{2}\eta^2]} \tag{67}$$

where

$$c_0 = -[\cos\alpha\cos\beta + (1+\tfrac{3}{2}\mu)\sin\alpha\sin\beta], \quad \mu = \frac{\lambda}{G}, \quad \eta = \frac{\sqrt{J_2}}{\sqrt{2}G},$$

and

$$a_1 = 2\eta + \eta^2(t_1+t_2)$$

$$a_2 = 2 + \mu + 2\eta t_2 + \eta^2 t_1 t_2$$

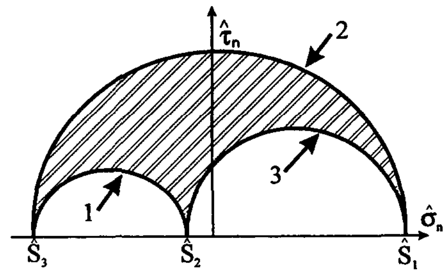

Fig. 1. The Mohr circles on the plane of the unit deviatoric normal and shear stress.

$$
\begin{aligned}
a_3 &= 2\eta^2 t_1 - \eta(1+\mu) \\
a_4 &= \tfrac{1}{2}\eta^2 - (1+\mu) + 2\eta^2 t_1 t_2 + \eta(3t_1 + t_2) \\
a_5 &= \tfrac{1}{2}\eta(2+\mu) + 3\eta t_1 t_2 + t_1(1 - \tfrac{1}{2}\eta^2) + t_2(1 + \tfrac{1}{2}\eta^2) + 2\eta^2 \det \hat{\mathbf{S}} \\
a_6 &= t_1 t_2(1 - \tfrac{1}{2}\eta^2) + 2t_2\eta^2 \det \hat{\mathbf{S}} + (2+\mu)\eta \det \hat{\mathbf{S}}
\end{aligned}
$$

$$
t_1 = \frac{1}{\sqrt{3}}(1 + \tfrac{2}{3}\mu)\tan\alpha, \quad t_2 = \frac{1}{\sqrt{3}}(1 + \tfrac{3}{2}\mu)\tan\beta.
$$

The function H in eqn (67) can be interpreted as a surface over the plane $(\hat{\sigma}_n, \hat{\tau}_n)$ and corresponds to the localization condition for the general three dimensional case. The stress states in this function are restricted to the dashed area limited by the Mohr circles (Fig. 1). These Mohr circles are given by:

$$
(1) \quad \hat{\tau}_n^2 + \left(\hat{\sigma}_n - \frac{\hat{S}_2 + \hat{S}_3}{2}\right)^2 = \left(\frac{\hat{S}_2 - \hat{S}_3}{2}\right)^2
$$

$$
(2) \quad \hat{\tau}_n^2 + \left(\hat{\sigma}_n - \frac{\hat{S}_1 + \hat{S}_3}{2}\right)^2 = \left(\frac{\hat{S}_1 - \hat{S}_3}{2}\right)^2
$$

$$
(3) \quad \hat{\tau}_n^2 + \left(\hat{\sigma}_n - \frac{\hat{S}_1 + \hat{S}_2}{2}\right)^2 = \left(\frac{\hat{S}_1 - \hat{S}_2}{2}\right)^2,
$$

or in a common form

$$
\hat{\tau}_n^2 + \hat{\sigma}_n^2 + \hat{\sigma}_n \hat{S}_i + \hat{S}_i^2 - \tfrac{1}{2} = 0, \tag{68}
$$

where i is 1, 2 and 3 for the first, the second and the third Mohr circle, respectively. The ranges of variation of the principal components of $\hat{\mathbf{S}}$ according to eqn (66) are as follows: $1/\sqrt{6} \leqslant \hat{S}_1 \leqslant \sqrt{2/3}$, $-1/\sqrt{6} \leqslant \hat{S}_2 \leqslant 1/\sqrt{6}$, $-\sqrt{2/3} \leqslant \hat{S}_3 \leqslant -1/\sqrt{6}$.

The task is to find the critical plastic hardening modulus by means of the maximum of H above the dashed closed area. The necessary conditions for a stationary value of the function H in eqn (67) are:

$$
\frac{\partial(H/2G)}{\partial\hat{\tau}_n} = 0, \quad \frac{\partial(H/2G)}{\partial\hat{\sigma}_n} = 0. \tag{69}
$$

From eqn $(69)_1$ we obtain:

$$
(\eta\hat{\sigma}_n + \eta t_2 + 1 + \tfrac{1}{2}\mu)(\eta\hat{\sigma}_n + 1 - \eta\hat{S}_1)(\eta\hat{\sigma}_n + 1 - \eta\hat{S}_2)(\eta\hat{\sigma}_n + 1 - \eta\hat{S}_3) = 0. \tag{70}
$$

It is easy to check that for the stress levels which are of practical interest $\eta \ll 1$ (or $G/\sqrt{3J_2} > 1$), the four roots in eqn (70), $\hat{\sigma}_{ni}$, $i = 1, 2, 3, 4$ are located outside the closed interval $[\hat{S}_3, \hat{S}_1]$. Consequently, the maximum of H in question must be above the boundary of the dashed area in Fig. 1, that is above a set restricted to the Mohr circles (68). Substituting eqn (68) into H eqn (64), via the elimination of $\hat{\tau}_n$, and going ahead with algebraic simplifications, we obtain the following three functions ($i = 1, 2, 3$):

$$\frac{H_i}{2G} = c_0 + \frac{2\cos\alpha\cos\beta(A_i\hat{\sigma}_n^2 + B_i\hat{\sigma}_n + C_i)}{(2+\mu)(D\hat{\sigma}_n + E_i)}, \tag{71}$$

where

$$\left.\begin{aligned} A_i &= 2\eta t_1 - 1 - \mu - \eta\hat{S}_i \\ B_i &= 2\eta(\tfrac{1}{2} - \hat{S}_i^2) - \hat{S}_i(2+\mu) + 2\eta t_1 t_2 + \eta\hat{S}_i(t_1 - t_2) + t_1 + t_2 \\ C_i &= (\tfrac{1}{2} - \hat{S}_i^2)(2 + \mu + 2\eta t_2) + t_1 t_2(1 + \eta\hat{S}_i) \\ D &= 2\eta \\ E_i &= 1 + \eta\hat{S}_i. \end{aligned}\right\} \tag{72}$$

In Appendix B, it is proved that:

$$H_2(\hat{\sigma}_n) > H_j, j = 1, 3, \quad \forall\hat{\sigma}_n \in (\hat{S}_3, S_1).$$

Thus, the stationary value of normal stress $(\hat{\sigma}_n)_m$ is determined by $\partial(H_2/2G)/(\partial\hat{\sigma}_n) = 0$, which results in:

$$(\hat{\sigma}_n)_m = \frac{-A_2E_2 - \sqrt{A_2(A_2E_2^2 + C_2D^2 - B_2DE_2)}}{A_2D}. \tag{73}$$

The corresponding maximum value of H_m is obtained from eqn (71):

$$\frac{H_m}{2G} = c_0 + \frac{2\cos\alpha\cos\beta}{(2+\mu)D}(2A_2(\hat{\sigma}_n)_m + B_2). \tag{74}$$

Let $\mathbf{n}$ be the unit normal to the surface of the shear band and let θ denote the angle in the x_1, x_3 plane from the x_1-axis to the normal vector $\mathbf{n}(\cos\theta, \sin\theta)$ or the angle between the x_3-axis and the shear band. In this case, the shear band normal vector lies in one of the planes formed by two of the principal axes of stress. Then, from eqn $(40)_1$:

$$\begin{aligned} \hat{\sigma}_n &= \hat{S}_1\cos^2\theta + \hat{S}_3\sin^2\theta = \tfrac{1}{2}(\hat{S}_1 + \hat{S}_3) + \tfrac{1}{2}(\hat{S}_1 - \hat{S}_3)\cos 2\theta = \\ &\quad -\tfrac{1}{2}\hat{S}_2 + \tfrac{1}{2}\cos 2\theta\sqrt{2 - 3\hat{S}_2^2} = p_1 + p_2\cos 2\theta = p_1 - p_2 + \frac{2p_2}{1+\tan^2\theta}. \end{aligned} \tag{75}$$

Upon the substitution of equation (73) into (75), the solution for the corresponding orientation of the band is given by:

$$\tan^2\theta_m = \frac{p_2 + p_1 - (\hat{\sigma}_n)_m}{p_2 - p_1 + (\hat{\sigma}_n)_m}. \tag{76}$$

It is important to note that H_m in eqn (74) is a function of the actual value of the stress level through J_2. In order to evaluate the critical shear band orientation and the critical plastic hardening modulus, we need to consider a specific material function of $H(J_2)$ which should be equalized to the function H_m eqn (74) and solved for J_2. This value can be substituted into eqn (73) to yield the critical normal stress, and from eqns (74) and (76) we obtain the corresponding value of the critical hardening modulus and critical shear band orientation, respectively.

In addition, for a zero stress substitution $\eta = 0$, the presented expressions are reduced to forms, the result of which is known from the literature for the small deformation case (see e.g. Runesson *et al.*, 1991; Bardet, 1991; Bigoni and Hueckel, 1991).

Remark 7

It may be of interest to evaluate eqn (77) for the incompressible Mises solids. In this case $\mu \to \infty$ and $\alpha = \beta = t_1 = t_2 = 0$, thus eqn (77) reduces to:

$$\tan^2 \theta_m = \sqrt{\frac{1+\eta\sqrt{2-3\hat{S}_2^2}}{1-\eta\sqrt{2-3\hat{S}_2^2}}}. \tag{77}$$

For the case of axially-symmetric tension ($\hat{S}_2 = 1/\sqrt{6}$) or compression ($\hat{S}_2 = -1/\sqrt{6}$), from eqn (77):

$$\tan^2 \theta_m = \sqrt{\frac{2G+\sqrt{3J_2}}{2G-\sqrt{3J_2}}}. \tag{78}$$

This is the critical shear band orientation as derived by Hutchinson and Tvergaard (1981).

In case of pure shear ($\hat{S}_2 = 0$), using eqn (77), the shear band orientation is given by:

$$\tan^2 \theta_m = \sqrt{\frac{2G+\sqrt{J_2}}{2G-\sqrt{J_2}}}. \tag{79}$$

Note that, for compressible Mises solids in the case of pure shear, the shear band orientation has the same form as expressed in eqn (79). □

Finally, it can be concluded that for the Jaumann–Cauchy formulation the vector normal to the plane of localization is perpendicular to the $\hat{S}_2$-direction. We note that, in the case of small deformation (neglecting the co-rotational effect), for the same constitutive model, it is easy to prove this proposition as was treated by Rudnicki and Rice (1975) and Benallal (1992).

5. CONCLUSIONS

Some explicit expressions of the plastic hardening modulus have been presented for a general nonassociative flow rule in the case of large deformations. The models on which these are based use three well-known objective stress rate formulations. In addition, the J_2 corner theory combined with the Jaumann rate of the Kirchhoff stress tensor has been discussed. In many practical cases the second- and third-order terms of the stress components in these expressions are neglected. However, here this is not so and we can satisfactorily treat other cases. One example is when the critical hardening modulus is of the same order as or smaller than the initial stress level, another is when the instability develops at a very small negative slope of the stress–strain diagram, for a strain-softening type localization, which occurs very close to the peak stress point (see Bazant, 1988).

Because complete and general expressions for the acoustic tensors are presented it is possible to use them as a starting point for comparing the models with respect to acceleration waves and other studies.

In the second part of this paper, the explicit expressions for the shear band orientation and the plastic hardening modulus are given for the Jaumann–Cauchy formulation, which depends on the magnitude of the stress only through the simple parameter J_2. These expressions are valid for a deviatoric associative flow rule, and it assumed that the stress tensor and the unit outward to the plastic potential and yield surface are coaxial.

Moreover, it has been shown that in the case of the Jaumann–Cauchy formulation the vector normal to the critical plane of localization is perpendicular to the direction of the second component of the unit deviatoric stress $\hat{S}_2$ expressed in eqn (66).

Finally, it may be worth noting that the present formalism can be used as a support for developing finite element programs that cover the shear band analysis in the large deformation range. For the two-dimensional cases, the closed form expressions presented here, eqns (74) and (76) can be used directly in a FEM code validation.

Acknowledgements—This work was written during a stay of the author in Japan as an Associate Professor at Yamaguchi University. The hospitality of the Japanese Ministry of Education and Professor Koji Nakagawa, Yamaguchi University, is gratefully appreciated. Several helpful discussions with Professor Chikayoshi Yatomi, Kanazawa University (Japan) and Professor Finn Ouchterlony, Swedish Detonic Research Foundation (Sweden), are also gratefully acknowledged. Special thanks are also directed to Professor Hussein M. Zbib, Washington State University (U.S.A.) for his advice.

REFERENCES

Asaro, R. J. and Rice, J. R. (1977). Strain localization in ductile single crystals. *J. Mech. Phys. Solids* **25,** 309–338.

Bardet, J. P. (1990). A comprehensive review of strain localization in elastoplastic soils. *Computers Geotech.* **10,** 163–188.

Bardet, J. P. (1991). Analytical solutions for the plane-strain bifurcation of compressible solid. *J. appl. Mech.* **58,** 651–657.

Bazant, Z. P. (1988). Softening Instability: Part I—Localization Into a Planar Band. *J. appl. Mech.* **55,** 517–522.

Benallal, A. (1992). On localization phenomena in thermo-elasto-plasticity. *Archs. Mech.* **44,** 15–29.

Bigoni, D. and Hueckel, T. (1990). On uniqueness and strain localization in plane strain and plane stress elastoplasticity. *Mech. Res. Comm.* **17,** 17–23.

Bigoni, D. and Hueckel, T. (1991). Uniqueness and localization—I. Associative and nonassociative elasto-plasticity. *Int. J. Solids Struct.* **28,** 197–213.

Bigoni, D. and Hueckel, T. (1993). On strain localization analysis of elastoplastic materials at finite strain. *Int. J. Plasticity* **9,** 21–33.

Christoffersen, J. and Hutchinson, J. W. (1979). A class of phenomenological corner theories of plasticity. *J. Mech. Phys. Solids* **27,** 465–487.

Dafalias, Y. F. (1983). Corotational rates for kinematic hardening at large plastic deformations. *J. appl. Mech.* **50,** 561–565.

Dafalias, Y. F. (1985a). A missing link in the macroscopic constitutive formulation of large plastic deformations. In *Plasticity Today: Modelling, Methods and Applications, Proceedings of the Int. Symp. on Current Trends and results in Plasticity*, CISM, Udine, 27–30 June, 1983 (Edited by Sawczuk, A. and Bianchi, G.), pp. 135–151. Elsevier, London.

Dafalias, Y. F. (1985b). The plastic spin. *J. appl. Mech.* **52,** 865–871.

Dafalias, Y. F. and Aifantis, E. C. (1990). On the microscopic origin of the plastic spin. *Acta Mech.* **82,** 31–48.

Duszek, M. K. and Perzyna, P. (1991). The localization of plastic deformation in thermo-plastic solids. *Int. J. Solids Struct.* **27,** 1419–1443.

Hadamard, J. (1903). *Leqons sur la Propagation des Ondes et les Equations de l'Hydro-dynamique*. Hermann, Paris.

Hill, R. (1962). Acceleration waves in solids. *J. Mech. Phys. Solids* **10,** 1–16.

Hill, R. and Hutchinson, J. W. (1975). Bifurcation phenomena in the plane tensil test. *J. Mech. Phys. Solids* **23,** 239–264.

Hutchinson, J. W. and Tvergaard, V. (1981). Shear band formulation in plane strain. *Int. J. Solids Struct.* **17,** 451–470.

Loret, B. (1983). On the effects of plastic rotation in the finite deformation of anisotropic elastoplastic materials. *Mech. Mater.* **2,** 287–304.

Loret, B. (1992). Does deviation from deviatoric associativity lead to the onset of flutter instability? *J. Mech. Phys. Solids* **40,** 1363–1375.

Loret, B. and Harireche, O. (1991). Acceleration waves, flutter instabilities and stationary discontinuities in inelastic porous media. *J. Mech. Phys. Solids* **39,** 569–606.

Loret, B., Prévost, J. H. and Harireche, O. (1990). Loss of hyperbolicity in elastic–plastic solids with deviatoric associativity. *Eur. J. Mech., A/Solids* **9,** 225–231.

Mandel, J. (1966). Conditions de Stabilit et Postulat de Drucker. In *Rheology and Soil Mechanics* (Edited by Kravtchenko, J. and Sirieys, P. M.), pp. 58–68. Springer, Berlin.

Mear, M. E. and Hutchinson, J. W. (1985). Influence of yield surface curvature on flow localization in dilatant plasticity. *Mech. Mater.* **4,** 395–407.

Molenkamp, F. (1985). Comparison of frictional material models with respect to shear band initiation. *Geotechnique* **35,** 127–143.

Needleman, A. (1979). Non-normality and bifurcation in plane strain tension and compression. *J. Mech. Phys. Solids* **27,** 231–254.

Needleman, A. and Ortiz, M. (1991). Effect of boundaries and interfaces on shear-band localization. *Int. J. Solids Struct.* **28,** 859–877.

Needleman, A. and Rice, J. R. (1978). Limits to ductility set by plastic flow localization. In *Mechanics of Sheet Metal Forming* (Edited by Koistinen, D. P. and Wang, N. M.), pp. 237–267. Plenum Press, New York.

Ottosen, N. S. and Runesson, K. (1991a). Properties of discontinuous bifurcation solutions in elasto-plasticity. *Int. J. Solids Struct.* **27,** 401–421.

Ottosen, N. S. and Runesson, K. (1991b). Acceleration waves in elasto-plasticity. *Int. J. Solids Struct.* **28**, 135–159.

Paulun, J. E. and Pecherski, R. B. (1987). On the application of the plastic spin concept for the description of anisotropic hardening in finite deformation plasticity. *Int. J. Plasticity* **3**, 303–314.

Peirce, D. (1983). Shear band bifurcations in ductile single crystals. *J. Mech. Phys. Solids* **31**, 133–153.

Prévost, J. H. (1984). Localization of deformations in elastic-plastic solids. *Int. J. numer. Anal. Methods Geomech.* **8**, 187–196.

Rice, J. R. (1976). The localization of plastic deformation. In *Theoretical and Applied Mechanics, Proc. 14th IUTAM Congress*, Delft, The Netherlands, 30 August–4 September 1976 (Edited by W. T. Koiter), pp. 207–220. North-Holland, Amsterdam.

Rice, J. R. and Rudnicki, J. W. (1980). A note on some features of the theory of localization of deformation. *Int. J. Solids Struct.* **16**, 597–605.

Rudnicki, J. W. and Rice, J. R. (1975). Conditions for the localization of deformation in pressure-sensitive dilatant materials. *J. Mech. Phys. Solids* **23**, 371–394.

Runesson, K., Ottosen, N. S. and Peric, D. (1991). Discontinuous bifurcations of elastic plastic solutions at plane stress and plane strain. *Int. J. Plasticity* **7**, 99–121.

Simo, J. C. (1985). On the computational significance of the intermediate configuration and hyperelastic relations in finite deformation plasticity. *Mech. Mater.* **4**, 439–451.

Simo, J. C. (1988). A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: Part I. Continuum formulation. *Comp. Meth. appl. Mech. Engng* **66**, 199–219.

Simo, J. C. and Pister, K. S. (1984). Remarks on rate constitutive equations for finite deformation problems: computational implications. *Comp. Meth. appl. Mech. Engng* **46**, 201–215.

Szabó, L. (1988). Discussion of "On constitutive relations at finite strain: hypo-elasticity and elasto-plasticity with isotropic or kinematic hardening, by Atluri, S. N." *Comp. Meth. appl. Mech. Engng* **67**, 125–127.

Szabó, L. (1992). Einige Bemerkungen zu der Arbeit von Wriggers, P.; Stein, E., "Die verwendung von Lie-Ableitungen bei der Angabe von Spannungsflussen für grosse Deformationen." *ZAMM* **72**(9), 451–452.

Szabó, L. and Balla, M. (1989). Comparison of some stress rates. *Int. J. Solids Struct.* **25**, 279–297.

Thomas, T. Y. (1961). *Plastic Flow and Fracture of Solids.* Academic Press, New York.

Tvergaard, V., Needleman, A. and Lo, K. K. (1981). Flow localization in the plane strain tensile test. *J. Mech. Phys. Solids* **29**, 115–142.

Tvergaard, V. and Van der Giessen, E. (1991). Effect of triaxial tension on flow localization for different plastic spins. In *Anisotropy and Localization of Plastic Deformation, Proc. of Plasticit 91: the third Int. Symp. on Plasticity and its Current Application*, Grenoble, France, 12–16 August, 1991 (Edited by J. P. Boehler and A. S. Khan), pp. 91–94. Elsevier Applied Science, London.

Van der Giessen, E. (1991). The plastic spin: micromechanical and thermodynamic aspects. *Int. J. Plasticity* **7**, 365–386.

Van der Giessen, E., Wu, P. D. and Neale, K. W. (1992). The effect of plastic spin on large strain elastic–plastic torsion of solid bars. *Int. J. Plasticity* **8**, 773–801.

Voyiadjis, G. Z. and Kattan, P. I. (1989). Eulerian constitutive model for finite deformation plasticity with anisotropic hardening. *Mech. Mater.* **7**, 279–293.

Yatomi, C., Yashima, A., Iizuka, A. and Sano, I. (1989). General theory of shear bands formation by a non-coaxial cam-clay model. *Soils Found.* **29**, 41–53.

Young, N. J. B. (1976). Bifurcation phenomena in the plane compression test. *J. Mech. Phys. Solids* **24**, 77–91.

Zbib, H. M. and Aifantis, E. C. (1988a). On the concept of relative spins and its implications to large deformation theories: I & II. *Acta Mech.* **75**, 15–33, and 35–56.

Zbib, H. M. and Aifantis, E. C. (1988b). On the localization and post-localization behavior plastic deformation. I. On the initiation of shear bands. *Res. Mech.* **23**, 261–277.

Zbib, H. M. (1989). Deformations of materials exhibiting noncoaxiality and finite rotations. *Scripta metall.* **23**, 789–794.

Zbib, H. M. (1991). On the mechanics of large inelastic deformations: noncoaxiality, axial effects in torsion and localization. *Acta Mech.* **87**, 179–196.

Zbib, H. M. (1993). On the mechanics of large inelastic deformations: kinematics and constitutive modeling. *Acta Mech.* **96**, 119–138.

Zhu, H., Zbib, H. M. and Aifantis, E. C. (1992). On the effect of anisotropy and inertia on shear banding: instability of biaxial stretching. *Appl. Mech. Rev.* **45**, 110–117.

APPENDIX A

The parameters in eqn (30):

$$d_4 = 4e_2^3 - e_1^2e_2^2 - 18e_1e_2e_3 + 4e_1^3e_3 + 27e_3^2$$

$$d_3 = 4cd_4 - D$$

$$d_2 = c(d_4 - 3D) + 5c^2d_4 + C$$

$$d_1 = 4c^3d_4 - 3c^2D + 2cC - B$$

$$d_0 = A - cB + c^2C - c^3D + c^4d_4,$$

where

$$c = \mathbf{P}{:}\mathbf{C}^e{:}\mathbf{Q}$$
$$A = p_1^2(4p_1p_3 - p_2^2)$$
$$B = 4p_2^3 - 18p_1p_2p_3 + 4p_1^2(e_3p_1 + 3e_1p_3) - 2p_1p_2(e_1p_2 + e_2p_1)$$
$$C = (e_1p_2 + e_2p_1)^2 + 12e_2p_2^2 - 18(e_1p_2p_3 + e_2p_1p_3 + e_3p_1p_2) + 12e_1p_1(e_3p_1 + e_1p_3) + 27p_3^2 + 2e_1e_2p_1p_2$$
$$D = 12e_2^2p_2 - 2e_1e_2(e_1p_2 + e_2p_1) - 18(e_1e_3p_2 + e_2e_3p_1 + e_1e_2p_3) + 4e_1^2(e_1p_3 + 3e_3p_1) + 54e_3p_3.$$

APPENDIX B

Proposition 2

Consider a simple form of function $H_i/2G$ in eqn (71) as:

$$\frac{H_i}{2G} = f(\hat{\sigma}_n, \hat{S}_i). \tag{B1}$$

Then

$$\frac{H_2}{2G} - \frac{H_1}{2G} \geqslant 0, \quad \forall \hat{\sigma}_n \in [\hat{S}_3, \hat{S}_2], \tag{B2a}$$

and

$$\frac{H_2}{2G} - \frac{H_3}{2G} \geqslant 0, \quad \forall \hat{\sigma}_n \in [\hat{S}_2, \hat{S}_1] \tag{B2b}$$

always hold.

Proof

Substituting the parameters (72) into (B1) and calculating the functions H_2 and H_1 in (B2a), after some algebraic manipulations we obtain:

$$\frac{H_2}{2G} - \frac{H_1}{2G} = 2\eta^2(\hat{S}_1 - \hat{S}_2)(\hat{\sigma}_n - \hat{S}_3)\left[\hat{\sigma}_n + t_2 + \frac{1}{2\eta}(2+\mu)\right]\left[\hat{\sigma}_n - \hat{S}_3 + \frac{1}{\eta}\right] \geqslant 0. \tag{B3}$$

Because of $\eta > 0$, $\hat{S}_1 - \hat{S}_2 \geqslant 0$ and $\hat{\sigma}_n - \hat{S}_3 \geqslant 0$, thus inequality (B3) reduces to:

$$F(\hat{\sigma}_n) = \left[\hat{\sigma}_n + t_2 + \frac{1}{2\eta}(2+\mu)\right]\left[\hat{\sigma}_n - \hat{S}_3 + \frac{1}{\eta}\right] > 0. \tag{B4}$$

It can be easily shown the quadratic function $F(\hat{\sigma}_n)$ in eqn (B4) is strictly positive $\forall \hat{\sigma}_n \in [\hat{S}_3, \hat{S}_2]$. Namely, the function $F(\hat{\sigma}_n)$ has a minimum and the two roots of this function are smaller than $\hat{S}_3$.

The inequality (B2b) can be proved in a similar way.